Rare K decays in the Standard Model

Christopher Smith*
Institut für Theoretische Physik, Universität Bern, CH-3012 Bern, Switzerland

The very clean theoretical predictions for the rare decays $K \to \pi\nu\bar{\nu}$ and $K_L \to \pi^0\ell^+\ell^-$ are reviewed, and their various theoretical inputs summarized. The less favorable situation for $K_L \to \mu^+\mu^-$ is also commented.

I. INTRODUCTION

The rare decays $K \to \pi\nu\bar{\nu}$ and $K_L \to \pi^0\ell^+\ell^-$, driven by semi-leptonic flavor-changing neutral currents (FCNC), are exceptionally clean probes of the flavor structure of the Standard Model, or of the still elusive New Physics. Concerted theoretical efforts have brought the SM predictions to an impressive level of accuracy (for the current experimental situation, see [1]). In this section, the main theoretical ingredients are briefly reviewed, while the situation for each mode is summarized in the following sections.

A. FCNC electroweak structure

FCNC arise at one loop in the electroweak theory. The processes driving the rare semi-leptonic K decays are the W box, Z and γ penguins[2], see Fig.1, and lead to the amplitudes

$$\mathcal{A}\left(K_L \to \pi^0 X\right) = \sum_{q=u,c,t} \left(\operatorname{Im}\lambda_q + \varepsilon \operatorname{Re}\lambda_q\right) y_q^X\left(m_q\right)\ ,$$
$$\mathcal{A}\left(K^+ \to \pi^+ X\right) = \sum_{q=u,c,t} \left(\operatorname{Re}\lambda_q + i \operatorname{Im}\lambda_q\right) y_q^X\left(m_q\right)\ ,$$

with $X = \nu\bar{\nu}, \ell^+\ell^-$ and $\lambda_q = V_{qs}^* V_{qd}$. In standard terminology, the ε part is the indirect CP-violating piece (ICPV), while the $\operatorname{Im}\lambda_q$ part is called direct CP-violating (DCPV). The amplitude for K_S is obtained from the K_L one by interchanging $\operatorname{Im}\lambda_q \leftrightarrow \operatorname{Re}\lambda_q$.

Without the dependence of the loop functions y_q^X on the quark masses, CKM unitarity would imply vanishing FCNC (GIM mechanism). Now, looking at these dependences, combined with the scaling of the CKM elements, one can readily get a handle on the importance of each quark contribution, and thereby on the cleanness achievable for the decay once QCD effects are included.

For $X = \nu\bar{\nu}$, only the Z penguin and W box enter, $y_q^{\nu\bar{\nu}} \sim m_q^2$, and light-quark contributions are suppressed. Since, in addition, $\varepsilon \sim 10^{-3}$ and $\operatorname{Re}\lambda_t \sim \operatorname{Im}\lambda_t$, ICPV is very small. For K^+, the c-quark contribution is suppressed from the loop, but enhanced by $\operatorname{Re}\lambda_c >> \lambda_t$, and ends up being comparable to the t-quark contribution.

For $X = \ell^+\ell^-$, the photon penguin also enters with its scaling $y_q^{\ell\ell} \sim \log(m_q)$ for $m_q \to 0$. In the standard CKM phase-convention, DCPV is still short-distance dominated thanks to $\operatorname{Im}\lambda_u = 0$, but not ICPV, completely dominated by the long-distance u-quark photon penguin, $K_1 \to \pi^0\gamma^* \to \pi^0\ell^+\ell^-$. The same holds for $K^+ \to \pi^+\ell^+\ell^-$, completely dominated by long-distance and therefore not very interesting for New Physics search.

For $K_L \to \ell^+\ell^-$, there is no photon penguin, and the electroweak structure is similar to $K \to \pi\nu\bar{\nu}$, up to the change $\operatorname{Im}\lambda_q \leftrightarrow \operatorname{Re}\lambda_q$.

Along with these contributions, there can be two-loop, third order electroweak contributions, if the extra suppression is compensated by non-perturbative long-distance enhancement. This occurs for modes with charged leptons, where the double-photon penguin gives a CP-conserving contribution ($\sim \operatorname{Re}\lambda_q$) to $K_L \to \pi^0\ell^+\ell^-$ and $K_L \to \ell^+\ell^-$, and is completely dominated by long-distance (u-quark), Fig.2c.

B. QCD corrections

Having identified the relevant electroweak structures, QCD effects have now to be included. This is done in three main steps:

Step 1: Integration of heavy degrees of freedom (top, W, Z), including perturbative QCD effects above M_W. This generates local FCNC operators (Fig.1 with t-quark), and Fermi-type four-fermion local operators.

Step 2: Resummation of QCD corrections (running down). At the c threshold (similar for b, τ), four-fermion operators are combined to form closed c-loops, which are then replaced by a tower of effective interactions in increasing powers of (external momentum)/(charm mass), Fig.2a. The lowest order consists again of the dimension-six FCNC operators, while dimension-eight operators are corrections scaling naively like $m_K^2/m_c^2 \sim 15\%$.

These first two steps (the OPE) can, in principle, be achieved to any desired level of precision within perturbative QCD, though the computation of the required multiloop diagrams represents a formidable task at higher orders. Still, this is unavoidable in order to reduce theoretical errors, in particular scale dependences. At this stage, one has obtained the complete Hamiltonian, i.e. all the effective operators, with the short-distance physics encoded in their Wilson coefficients.

Step 3: To get the amplitudes, the matrix elements of these operators between meson states remain to be

*Electronic address: chsmith@itp.unibe.ch

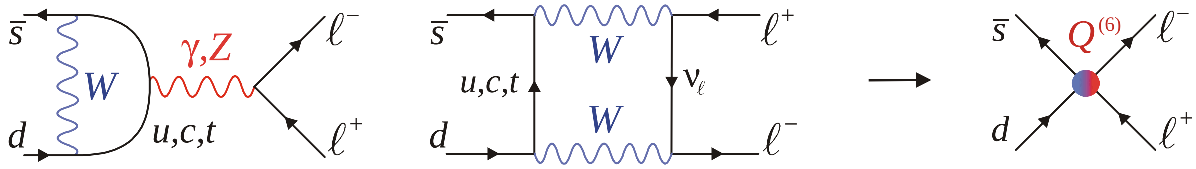

FIG. 1: The Z, γ penguin and W box generating the effective FCNC interactions relevant for rare K decays.

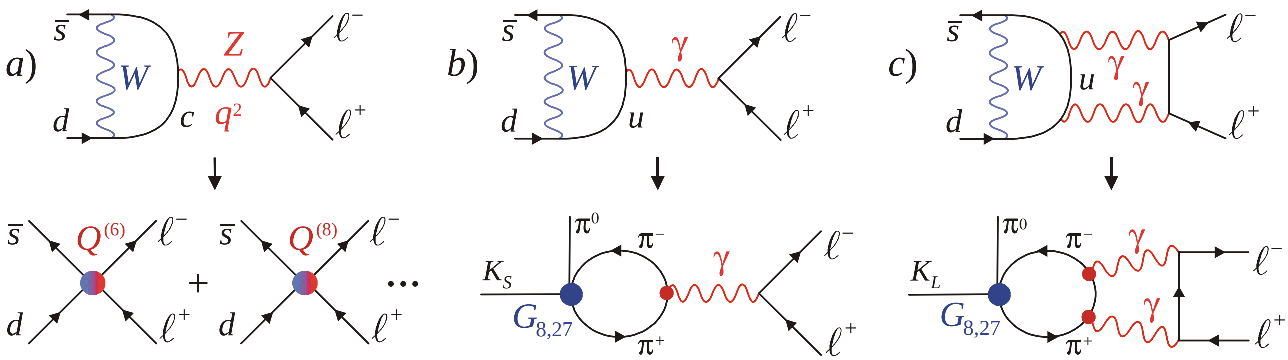

FIG. 2: Illustration of the treatment of various QCD effects. a) Tower of effective FCNC interactions generated by the c-quark integration. b) Non-local long-distance photon penguin with u-quark contribution represented by meson loops. c) Idem as b) for the double-photon penguin.

estimated. To this end, one makes use of the symmetries of QCD, as embodied in Chiral Perturbation Theory (ChPT), to relate the desired matrix elements to experimentally known quantities.

For the most interesting dimension-six semi-leptonic operators, the matrix elements can be related to those of $K_{\ell 2}, K_{\ell 3}$ decays (taking into account isospin-breaking corrections). Contributions from four light-quark operators ($Q_1, ..., Q_6$) are represented directly in terms of meson fields in ChPT, such that non-local u-quark loops are represented as meson loops (Fig.2b,c). The price to pay is the introduction of some unknown low-energy constants ($G_{8,27,...}$), to be extracted from experiment. In particular, G_8 is fixed from $\mathcal{B}(K \to \pi\pi)$, accounting for the large non-perturbative $\Delta I = 1/2$ effects. For dimension-eight operators, an approximate matching is done with the ChPT representation of the u-quark contributions.

II. THE $K^+ \to \pi^+ \nu\bar{\nu}$ AND $K_L \to \pi^0 \nu\bar{\nu}$ DECAYS

Thanks to the suppression of light-quark effects, these modes are the cleanest and their rates are precisely predicted within the SM. Their branching ratios read:

$$\mathcal{B}_+^{\nu\bar{\nu}} = \kappa_+ \left(\left| \frac{\operatorname{Im} \lambda_t}{\lambda^5} X_t \right|^2 + \left| \frac{\operatorname{Re} \lambda_t}{\lambda^5} X_t + \frac{\operatorname{Re} \lambda_c}{\lambda} P_{u,c} \right|^2 \right) ,$$

$$\mathcal{B}_L^{\nu\bar{\nu}} = \kappa_L \left| \frac{\operatorname{Im} \lambda_t}{\lambda^5} X_t \right|^2 ,$$

with $P_{u,c} = P_c + \delta P_{u,c}$. The Wilson coefficient of the dimension-six FCNC operator $Q^\nu = (\bar{s}d)_{V-A} (\bar{\nu}\nu)_{V-A}$ arising from the top-quark loop is known at NLO, $X_t = 1.646 \pm 0.041$[3]. For the charm-quark, the contribution to this operator has recently been obtained at NNLO[3], $P_c = 0.37 \pm 0.04$. Residual c-quark effects from dimension-eight operators[4], along with long-distance u-quark effects[5] amount to a small correction $\delta P_{u,c} = 0.04 \pm 0.02$[6]. The matrix elements of Q^ν are known from $K_{\ell 3}$, including the leading isospin-breaking corrections [7], and are encoded into $\kappa_L = 2.29 \pm 0.03 \cdot 10^{-10}$ and $\kappa_+ = 5.26 \pm 0.06 \cdot 10^{-11}$ for $\lambda = 0.225$. Finally, for $K_L \to \pi^0 \nu\bar{\nu}$, ICPV is of about 1%[8] while the CP-conserving contribution arising from box diagrams is less than 0.01%[9].

The SM predictions are then

$$\mathcal{B}\left(K_L \to \pi^0 \nu\bar{\nu}\right) = (2.7 \pm 0.4) \cdot 10^{-11} ,$$
$$\mathcal{B}\left(K^+ \to \pi^+ \nu\bar{\nu}\right) = (8.4 \pm 1.0) \cdot 10^{-11} .$$

The error on $K_L \to \pi^0 \nu\bar{\nu}$ is dominated by $\operatorname{Im} \lambda_t$, while for $K^+ \to \pi^+ \nu\bar{\nu}$, it breaks down to scales (13%), m_c(22%), CKM, α_S, m_t (37%) and matrix-elements from $K_{\ell 3}$ and light-quark contributions (28%)[3]. Further improvements are thus possible through a better knowledge of m_c, of the isospin-breaking in the $K \to \pi$ form-factors, or by a lattice study of higher-dimensional operators[10].

As the determination of λ_t from general UT fits to B physics data is already very precise, and expected to be further improved in the near future, the main interest of the $K \to \pi\nu\bar{\nu}$ decays is to test the CKM paradigm for CP-violation in the SM. Indeed, these modes do offer a

TABLE I: Coefficients encoding the various contributions to $\mathcal{B}^{\ell^+\ell^-}$

	C^ℓ_{dir}	C^ℓ_{int}	C^ℓ_{mix}	$C^\ell_{\gamma\gamma}$
$\ell = e$	$(4.62 \pm 0.24)\left(y_{7V}^2 + y_{7A}^2\right)$	$(11.3 \pm 0.3)\ y_{7V}$	$14.5 \pm 0.5,$	≈ 0
$\ell = \mu$	$(1.09 \pm 0.05)\left(y_{7V}^2 + 2.32 y_{7A}^2\right)$	$(2.63 \pm 0.06)\ y_{7V}$	3.36 ± 0.20	5.2 ± 1.6

particularly interesting independent determination since, as discussed in [11], they are very sensitive to a large class of New Physics models. As such, they also constitute one of the best windows into the flavor structure of the New Physics that will hopefully be uncovered at LHC.

III. THE $K_L \to \pi^0 \ell^+\ell^-$ DECAYS

Here the situation is more involved. The t and c-quark contributions generate both the dimension-six vector $Q_{7V} = (\bar{s}d)_V(\bar{\ell}\ell)_V$ and axial-vector $Q_{7A} = (\bar{s}d)_V(\bar{\ell}\ell)_A$ operators, whose Wilson coefficients $y_{7V,7A}$ are known to NLO[2]. The former produces the $\ell^+\ell^-$ pair in a 1^{--} state, the latter in a 1^{++} state and, in addition, for $\ell = \mu$, in a helicity-suppressed 0^{-+} state.

Indirect CP-violation is related to $K_S \to \pi^0\ell^+\ell^-$, for which the long-distance photon penguin dominates (Fig.2b). In ChPT, loops are small and one just needs to fix a counterterm, a_S[12]. This can be done up to a sign from NA48 measurements as $|a_S| = 1.2 \pm 0.2$[13]. Producing $\ell^+\ell^-$ in a 1^{--} state, it interferes with the contribution from Q_{7V}, arguably constructively[14, 15].

The CP-conserving (CPC) contribution from $Q_{1,\ldots,6}$ proceeds through two-photons, i.e. produces the lepton pair in either a helicity-suppressed 0^{++} or phase-space suppressed 2^{++} state. Only the 0^{++} state is produced at LO through the finite two-loop process $K_L \to \pi^0 P^+P^- \to \pi^0\gamma\gamma \to \pi^0\ell^+\ell^-$, $P = \pi, K$ (Fig.2c). Higher order corrections are estimated using $K_L \to \pi^0\gamma\gamma$ experimental data for both the 0^{++}[16] and 2^{++} contributions[14].

Altogether, the branching ratios are

$$\mathcal{B}^{\ell^+\ell^-} = (C^\ell_{dir} \pm C^\ell_{int}\,|a_S| + C^\ell_{mix}\,|a_S|^2 + C^\ell_{\gamma\gamma}) \cdot 10^{-12}\,,$$

with the coefficients given in Table I. Interestingly, these coefficients obey $C_i^\mu/C_i^e \approx 0.23$ due to the phase-space suppression, but for the helicity-suppressed contributions arising from the Q_{7A} operator (DCPV) and from $\gamma\gamma$ (CPC). This maintains the sensitivity of $\mathcal{B}^{\mu^+\mu^-}$ on the interesting short-distance physics at the same level as $\mathcal{B}^{e^+e^-}$. Further, it allows in principle to disentangle the Q_{7V} and Q_{7A} contributions from the measurements of both modes. This is illustrated in Fig.3a, where the hyperbola corresponds to a common rescaling of both y_{7A} and y_{7V}[16]. As discussed in [16, 17], this plane is particularly interesting to look for signals of New Physics, and identify its precise nature[11].

In the SM, $y_{7A}(M_W) = -0.68 \pm 0.03$ and $y_{7V}(\mu \approx 1\text{ GeV}) = 0.73 \pm 0.04$[2], and the predicted rates are[14, 16, 17]:

$$\mathcal{B}\left(K_L \to \pi^0 e^+e^-\right) = 3.54^{+0.98}_{-0.85}\ \left(1.56^{+0.62}_{-0.49}\right) \cdot 10^{-11}\,,$$
$$\mathcal{B}\left(K_L \to \pi^0 \mu^+\mu^-\right) = 1.41^{+0.28}_{-0.26}\ \left(0.95^{+0.22}_{-0.21}\right) \cdot 10^{-11}\,,$$

for constructive (destructive) interference (Fig.3a). Overall, the error on a_S is currently the most limiting and better measurements of $K_S \to \pi^0\ell^+\ell^-$ would be welcomed (Fig.3b). Also, better measurements of $K_L \to \pi^0\gamma\gamma$ would help reduce the error on the 0^{++} and 2^{++} contributions. Alternatively, they can be partially cut away through energy cuts or Dalitz plot analyses[14, 16, 17].

The integrated forward-backward (or lepton-energy) asymmetry [17, 18]

$$A^\ell_{FB} = \frac{N\left(E_{\ell^-} > E_{\ell^+}\right) - N\left(E_{\ell^-} < E_{\ell^+}\right)}{N\left(E_{\ell^-} > E_{\ell^+}\right) + N\left(E_{\ell^-} < E_{\ell^+}\right)}\,,$$

is generated by the interference between CP-conserving and CP-violating amplitudes. It cannot be reliably estimated at present for $\ell = e$ because of the poor theoretical control on the 2^{++} contribution. The situation is better for $\ell = \mu$, for which this part is negligible,

$$A^\mu_{FB} = (1.3\,(1)\,y_{7V} \pm 1.7(2)\,|a_S|) \cdot 10^{-12}\ /\ \mathcal{B}^{\mu^+\mu^-}\,,$$

i.e., $A^\mu_{FB,\mathrm{SM}} \approx 20\%(-12\%)$ for constructive (destructive) interference[17]. Interestingly, though the error is large, A^μ_{FB} can be used to fix the sign of a_S.

IV. THE $K_L \to \mu^+\mu^-$ DECAY

For this mode, the short-distance (SD) piece from t and c-quarks is known to NLO and NNLO[19], respectively. Indirect CP-violation is negligible. The long-distance (LD) contribution from $Q_{1,\ldots,6}$ matrix elements proceeds again through two-photons. Still, there are three differences with respect to the $K_L \to \pi^0\ell^+\ell^-$ decays.

First, the contribution from the imaginary part of the $\gamma\gamma$ loop, estimated from $K_L \to \gamma\gamma$, is much larger than SD, and already accounts for the bulk of the experimental $K_L \to \mu^+\mu^-$ rate. Second, while the charged meson loop in $K_L \to \pi^0\ell^+\ell^-$ acts like a cut-off, and a finite result is found, now the two photons arise from the axial anomaly, and $K_L \to \pi^0, \eta, \eta' \to \gamma\gamma \to \mu^+\mu^-$ is divergent, requiring unknown counterterms. To estimate them, though still with a large error, one can use the experimental information on $K_L \to \gamma^*\gamma^*$ together with the perturbative behavior of the $\bar{s}d \to \bar{u}u \to \gamma\gamma$ loop[20]. Finally, SD and LD produce the same 0^{-+} state and thus interfere. This

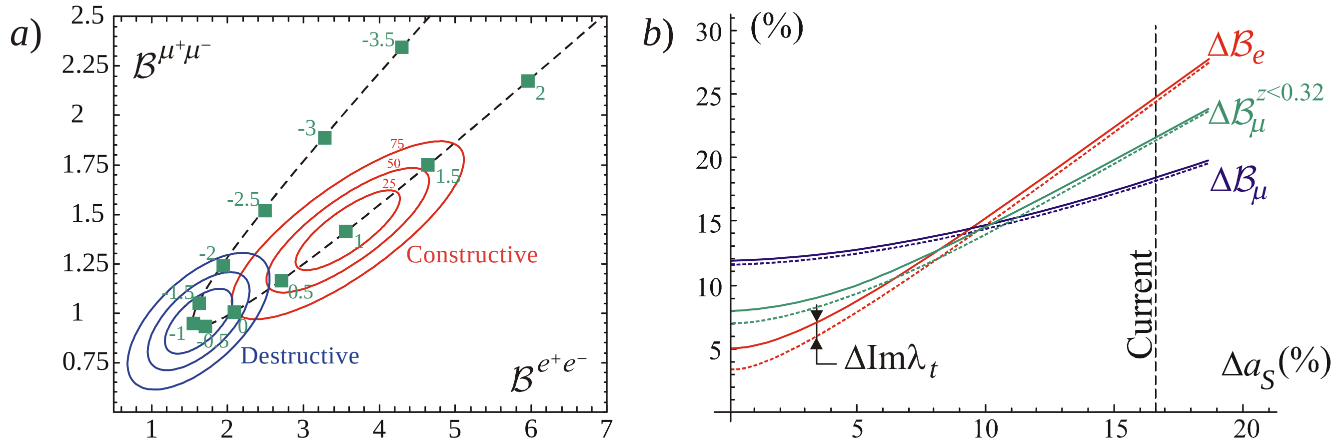

FIG. 3: a) $\mathcal{B}^{\mu^+\mu^-}$ against $\mathcal{B}^{e^+e^-}$, in units of 10^{-11}. The hyperbola denotes common rescalings of $y_{7A,7V}$ (or $\mathrm{Im}\,\lambda_t$), while the 25, 50 and 75% confidence regions correspond to the current SM predictions for constructive and destructive ICPV-DCPV interference. b) Evolution of the error on $\mathcal{B}^{\mu^+\mu^-}$ and $\mathcal{B}^{e^+e^-}$ as a function of the error on a_S. The residual error due to $\mathrm{Im}\,\lambda_t$ is smaller. The middle curve indicates the improvement achievable by selecting events with muon invariant-mass smaller than $2m_\pi$, which amounts to cutting away the bulk of the two-photon CPC contribution.

interference, which depends on the sign of $\mathcal{A}(K_L \to \gamma\gamma)$, is presumably constructive[21]. Better measurements of $K_S \to \pi^0\gamma\gamma$ or $K^+ \to \pi^+\gamma\gamma$ could settle this sign.

$K_L \to \mu^+\mu^-$ is thus obviously not as clean as $K \to \pi\nu\bar{\nu}$ or $K_L \to \pi^0\ell^+\ell^-$. Nevertheless, being measured precisely, it can still lead to interesting constraints in some specific scenarios like SUSY at large $\tan\beta$[11].

V. CONCLUSION

Thanks to the numerous theoretical efforts, the four rare decays, $K \to \pi\nu\bar{\nu}$ and $K_L \to \pi^0\ell^+\ell^-$, now provide for one of the cleanest and most sensitive tests of the Standard Model. These modes are promising not only to get clear signals of New Physics – or to severely constrain it –, but also to uncover the nature of the possible New Physics at play through the specific pattern of deviations they would exhibit with respect to the SM predictions.

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